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THEORETICAL ESTIMATION OF STRUCTURES: DIFFERENTIALLY ROTATING STELLAR MODELS OF NON-UNIFORM MASSES

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ABSTRACT

A method is proposed to compute equilibrium structures of differentially rotating polytropic stars. The law of differential rotation is assumed to be the form of $\omega^2(s) = b_1 + b_2 s^2 + b_3 s^4$ .The angular velocity of rotation (ω) is a function of distance (s) of a fluid element from the axis of rotation. Kippenhahn and Thomas's averaging approach and concepts of Roche-equipotential are utilised to incorporate the effects of differential rotation on the equilibrium structures of differentially rotating polytropic stellar models having non-uniform masses. The proposed mathematical modelling can be used to determine structures, oscillations and other distortions of pre-main sequence, main sequence and giant stars.

KEYWORDS: Polytropes, Roche-equipotential, equilibrium structure, tidal distortion, differential rotation, mass variation.

1. INTRODUCTION

Observations show that some variable stars are rotating about their centre of mass. In some cases these rotations are solid body rotations, however, others have differential rotation. In cases of differential rotation fluid particles can have different angular velocities. For a binary stellar system, primary component (more massive is generally considered as primary) generally remains larger in comparison to its secondary component. Stars of binary systems revolve around their common centre of mass and it is expected that these rotations can be uniformed or differential. It is also observed that differential rotations influence the inner structure and equilibrium configurations of such stars. It is also expected that the equilibrium structure of such a star in binary system is also influenced by the perturbations such as differential rotation, tidal distortion and mass variations etc.

In this paper, physical parameters related to the structures of differentially rotating gaseous spheres are expressed by using the law of differential rotation of the form $\omega^2(s) = b_1 + b_2 s^2 + b_3 s^4$, where $\omega(s)$ is angular velocity of rotation of a fluid element at distance s from the axis of rotation while b_1, b_2 and b_3 are numerical constants. Combinations of these arbitrary chosen constant give the different natures of rotating stellar models. Our techniques utilize the averaging approach of Kippenhahn and Thomas [4] and concepts of Roche-equipotential in a manner earlier used by Saini et al. [14] to incorporate the effect of differential rotation on the rotationally distorted stellar models. The inner structure of differentially rotating polytropic models with polytropic indices 1.5, 2.0, 3.0 and 4.0 can be approximated by using various expressions (defined in section 4.) of physical parameters with suitable combination of the parameters b_1, b_2 and b_3.

The exact mathematical study determining the problem of equilibrium structure of differentially rotating stars are quite complex. It becomes more complicated if stars are distorted by the effects such as tidal distortion, mass variation, Coriolis force and magnetic perturbations etc. A theory based on distorted polytropes was developed

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by Chandrasekhar [1]. Since then several authors such as Clement [2], Kopal [3], Lal et al. [13], Kumar et al. [10], Saini et al. [15, 16, 17] have addressed themselves to these problems. Kopal [6], Mohan and Singh [5], Saini et al. [14], Mohan et al. [7, 8 and 9] have observed that the actual equipotential surfaces of a rotationally and tidally distorted star are approximated by equivalent rotationally and tidally distorted Roche-equipotentials. Lal et al. [11 and 12] have applied this approach on white dwarf and polytropic stars and hence found their equilibrium structures. In this approximation, averaging approach of Kippenhahn and Thomas [4] and results of the Roche-equipotentials obtained by Kopal [6] are used to incorporate the rotational and tidal effects up to second order of smallness in the stellar structure equations. Once the Roche-equipotential surfaces of a differentially rotating star are approximated by modified Roche-equipotential, the approach used by Saini et al. [14], may now be used to evaluate explicitly the values of modified physical parameter $S_\psi, V_\psi, \overline{g}$ and $\overline{g}^{-1}$.

2. PROPOSED LAW OF DIFFERENTIAL ROTATION

The law of differential rotation is assumed in the form

$$\omega^2(s) = b_1 + b_2 s^2 + b_3 s^4, \tag{1}$$

where $s = r\sin\theta$ is a non-dimensional measure of the distance of a fluid element from the axis of rotation passing through its centre. This law can also be obtained by the expansion of Taylor series of this form $\omega^2 = f(s^2)$ up to second order of smallness. This law can generate a variety of commonly expected natures of different differentially rotating stars, and is also in a form which it can be conveniently subjected to the type of mathematical analysis that carried in this paper.

3. ROCHE-EQUIPOTENTIALS OF DIFFERENTIALLY ROTATING GASEOUS SPHERES INFLUENCED THE EFFECT OF MASS VARIATION

The concepts of Roche-equipotential and Roche limit have often been used in literature to investigate the physical parameters and stability of binary stars. While computing Roche-equipotentials, the whole mass of the sphere is assumed to be concentrated at its centre. This approximation, through reasonably correct for highly centrally condensed stellar models, but is not true for the stars which are not highly condensed at the centre. The concept of Roche-equipotentials, therefore, needs to be modified in case of the stars which are not highly centrally condensed, taking into account the effect mass variation will have on its equipotentials surfaces inside the star. Results on Roche-equipotential based on this modification are of practical interest in the present study.

Let M_0 is the entire mass of the differentially rotating primary component and it is more massive than its companion star of mass M_1 (i.e., $M_0 > M_1$). Suppose R is the mutual separation between two masses and the position of these components of binary system is referred to a rectangular system of Cartesian co-ordinates which have the origin at the centre of gravity of mass M_0, at x axis along the line joining the centres of the components, z axis perpendicular to the plane of the orbit of the two components, $M_0(r)$ is the interior mass of the primary component. Due to the gravitational, rotational and other disturbing forces acting an arbitrary point $P(x, y, z)$, total potential Ω may be expressed as

$$\Omega = G\frac{M_0(r)}{r} + G\frac{M_1}{r_1} + \frac{1}{2}\omega^2\left((x - \frac{M_1 R}{M_0 + M_1})^2 + y^2\right), \tag{2}$$

where $r^2 = x^2 + y^2 + z^2$ and $r_1^2 = (R - x)^2 + y^2 + z^2$.

In this modelling, distances of point p from the centres of primary and secondary are represented through r and r_1 respectively. Total potential Ω is the sum of potential arising from the mass of the component of mass M_0, disturbing potential of its companion of mass M_1 and potential arising from the centrifugal force.

The angular velocity ω is identical with the Keplarian angular velocity in a close binary system and can be defined as $\omega^2 = G(M_0 + M_1)/R^3$.

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On using relations (1) in (2) adopt as M_0 our unit mass, R as unit of length and choose the unit of time such that $G = 1$, equation (2) may be expressed in terms of polar spherical coordinates:

$$x = r\cos\phi\sin\theta = r\lambda,\ y = r\sin\phi\sin\theta = r\mu,\ z = r\cos\theta = r\nu. \qquad (3)$$

as
$$\psi = \frac{1}{r} + q\left[\frac{1}{\left(1 - 2\lambda r + r^2\right)^{1/2}} - \lambda r\right] + \frac{1}{2}\omega^2 r^2\left(1 - \nu^2\right), \qquad (4)$$

where $\psi = \frac{R\Omega}{GM_0} - \frac{M_1^2}{2M_0(M_0 + M_1)}$, $q = \frac{M_1}{M_0}$ and $t = \frac{M_0(r)}{M_0}$ are non-dimensional parameters and ω^2 is non-dimensional unit of GM_0/R^3.

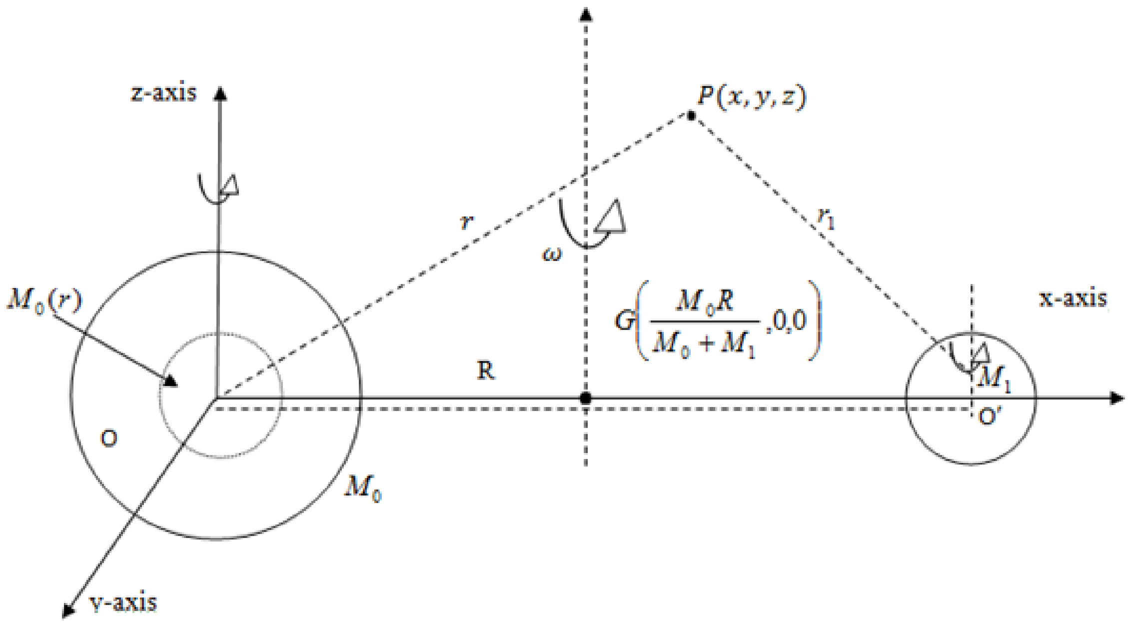

Fig. 1: Axes of reference

Surfaces generated by setting ψ =constant, are represented by left hand side of equation (4) and referred to as Roche-equipotentials. It approximates the equipotential surfaces of a star. Roche-equipotential depends entirely upon the value of ψ and corresponding equipotentials consists two separate ovals, if ψ is large, closed around each of the two mass points.

The right hand side of equation (4) can be large only if r or $r_1 = \left(1 - 2\lambda r + r^2\right)^{1/2}$ becomes small; it must be nearly equal to r and r_1, if the right hand side of equation (4) is to be constant. Therefore, large values of ψ correspond to equipotentials which differ little but spheres. With reduction in the value of ψ in the expression (4), the ovals define by expression (4) become increasingly elongated in the direction of the centre of gravity of the system until for a certain critical value of ψ (characteristic of each mass ratio) is achieved, both ovals will unite in a single point on the x-axis to form a dumb-bell like configuration. These limiting value of ψ are called Roche limits. Two ovals of their Roche limit filling by any pair of stars are called contact binaries. The connecting part of dumb-bell open up for smaller values of ψ and the corresponding equipotential surfaces envelope both the bodies.

ISSN: 2277-9655
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CODEN: IJESS7

4. PHYSICAL PARAMETERS S_ψ, V_ψ, $\overline{g}$ AND $\overline{g^{-1}}$ INCLUDING MASS VARIATION

Expressions to calculate volume, surface area and other physical parameters of differentially rotating polytropic models of stars are investigated in this section. For this, the approach is used as earlier adopted by Saini et al. [14] for obtaining the equilibrium structure of differentially rotating and tidally distorted Prasad models including the effect of mass variation inside the star, and Lal et al. [13] for obtaining the equilibrium structures of polytropic stars having differential rotation. To compute distortion effects, actual equipotentials surfaces of stars are approximated by Roche-equipotentials and Kopal's [6] result on the Roche-equipotentials are then used to express the problem in a form convenient for numerical work. In order to introduce the concept of Roche-equipotentials, assume a mass *M* and radius *R*, for rotating configuration, the total potential Ω of a fluid element is given by the equation of hydrostatic equilibrium may be written in the form:

$$d\Omega = dV + \frac{1}{2}\omega^2 d(s)^2$$

$$\text{or } \Omega = V + \int \omega^2(s) s ds$$

$$\text{i.e.} \quad \Omega = \frac{GM_0(r)}{r} + \int \omega^2(s) s ds \tag{5}$$

If a differentially rotating gaseous sphere is considered as Roche-model, gravitational potential at an arbitrary point $P(x, y, z)$ will be calculated by $V = GM_0(r)/r$, where $M_0(r)$ is mass interior to sphere of radius *r* and M_0 is the total mass of the rotating gaseous sphere.

On substituting it in (5) and multiplying throughout by R/GM_0, it can be reduced as

$$\psi = \frac{t}{r/R} + \frac{1}{2}\frac{R}{GM_0}\omega^2 d(s^2) \text{ with } t = \frac{M_0(r)}{M_0}. \tag{6}$$

Since dimension of s is same as that of *R*, assuming ω^2 to have a dimension of GM_0/R^3, the non-dimensional form of (6) can be represented as:

$$\psi = \frac{t}{r} + \frac{1}{2}\int \omega^2 d(s^2) . \tag{7}$$

Using $\omega^2(s) = b_1 + b_2 s^2 + b_3 s^4$ with $s^2 = r^2(1-\nu^2)$ in (7), it reduced as:

$$\psi = \frac{t}{r} + \frac{1}{2}b_1^2 r^2(1-\nu^2) + \frac{1}{2}b_1 b_2 r^4(1-\nu^2)^2 + \frac{1}{6}(2b_1 b_3 + b_2^2) r^6 (1-\nu^2)^3 + \frac{1}{4}b_2 b_3 r^8(1-\nu^2)^4 + \frac{1}{10}b_3^2 r^{10}(1-\nu^2)^5. \tag{8}$$

In absence of rotation $(b_1 = b_2 = b_3 = 0)$, Roche-equipotential (8) reduced to $\psi = t/r$ and in case of solid body rotation $(b_2 = b_3 = 0,\ t = 1)$, equation (8) reduced to the expression given by Mohan and Singh [5]. Now, ψ is the non-dimensional form of the total potential Ω ($\psi = R\Omega/GM$), $\lambda = \sin\theta, \sin\phi$, $\mu = \cos\theta$, $(r,\ \theta,\ \phi)$ being the polar spherical coordinates of the point with centre of the star as the origin, *x*- axis in the equatorial plane, θ being measured from *z*- axis.

Following Kopal [6] with the assumption $\psi =$ constant, one coordinate associated to the Roche-equipotential, is obtained as:

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$$r = r_0 R\left[1+\frac{1}{2t}b_1 r_0^3 x+\frac{1}{4t}b_2 r_0^5 x^2+\frac{3}{4t^2}b_1^{\,2} r_0^6 x^2+\frac{1}{6t}b_3 r_0^7 x^3+\frac{1}{t^2}b_1 b_2 r_0^8 x^3+\frac{3}{8t^3}b_1^3 r_0^9 x^3 + \frac{5}{48t^2}\left(8b_1 b_3+3b_2^2\right)r_0^{10}x^4+\ldots\right], \quad (9)$$

where $r_0 = t/\psi$ and $x = \left(1-\nu^2\right)$. In above equation, terms of t, b_1, b_2 and b_3 are retained up to third order of smallness. The shapes of various equipotential surfaces of differentially rotating gaseous spheres have obtained by setting r = constant. In equation (9) R is the radius of undistorted model.

Following Kopal [6] Mohan et al. [7] and Lal et al. [13], the explicit expression for S_ψ, V_ψ, $\overline{g}$ and $\overline{g}^{-1}$ are obtain as:

$$V_\psi = \frac{4\pi}{3}r_0^3 R^3\left[1+\frac{1}{t}b_1 r_0^3+\frac{2}{5t}b_2 r_0^5+\frac{8}{5t^2}b_1^2 r_0^6+\frac{8}{35t}b_3 r_0^7+\frac{12}{7t^2}b_1 b_2 r_0^8+\frac{8}{5t^3}b_1^3 r_0^9 + \frac{16}{105t^2}\left(8b_1 b_3+3b_2^2\right)r_0^{10}+\ldots\right]. \quad (10)$$

Following the averaging technique of Kippenhahn and Thomas [4] for a topologically equivalent sphere of radius r_ψ, we have the relation:

$$V_\psi = \frac{4}{3}\pi r_\psi^3. \quad (11)$$

On inserting (10) in (11)

$$r_\psi = r_0 R\left[1+\frac{1}{3t}b_1 r_0^3+\frac{2}{15t}b_2 r_0^5+\frac{19}{45t^2}b_1^2 r_0^6+\frac{8}{105t}b_3 r_0^7+\frac{152}{315t^2}b_1 b_2 r_0^8+\frac{97}{405t^3}b_1^3 r_0^9 + \left(\frac{16}{45t^2}b_1 b_3+\frac{212}{1575t^2}b_2^2\right)r_0^{10}+\ldots\right]. \quad (12)$$

The surface area of Roche-equipotentials surface ψ = constant is given by:

$$S_\psi = 4\pi r_0^2 R^2\left[1+\frac{2}{3t}b_1 r_0^3+\frac{4}{15t}b_2 r_0^5+\frac{14}{15t^2}b_1^2 r_0^6+\frac{16}{105t}b_3 r_0^7+\frac{36}{35t^2}b_1 b_2 r_0^8 + \frac{34}{35t^3}b_1^3 r_0^9+\frac{88}{945t^2}\left(8b_1 b_3+3b_2^2\right)r_0^{10}+\ldots\right]. \quad (13)$$

The explicit expressions of gravity $\overline{g}$ and its inverse $\overline{g^{-1}}$ can be shown to be respectively.

$$\overline{g} = \frac{tGM_\psi}{r_0^2 R^2}\left[1-\frac{4}{3t}b_1 r_0^3-\frac{4}{5t}b_2 r_0^5-\frac{7}{9t^2}b_1^2 r_0^6-\frac{64}{105t}b_3 r_0^7-\frac{488}{315t^2}b_1 b_2 r_0^8-\frac{134}{945t^3}b_1^3 r_0^9 - \frac{16}{4725t^2}\left(505\,b_1\,b_3+162b_2^2\right)r_0^{10}+\ldots\right]. \quad (14)$$

and

$$\overline{g^{-1}} = \frac{r_0^2 R^2}{tGM_\psi}\left[1+\frac{4}{3t}b_1 r_0^3+\frac{4}{5t}b_2 r_0^5+\frac{131}{45t^2}b_1^2 r_0^6+\frac{64}{105t}b_3 r_0^7+\frac{1352}{315t^2}b_1 b_2 r_0^8 + \frac{1198}{189t^3}b_1^3 r_0^9+\frac{16}{4725t^2}\left(1145b_1 b_3+432b_2^2\right)r_0^{10}+\ldots\right]. \quad (15)$$

In the above expressions, M_ψ is the mass contained with Roche- equipotentials.

5. CONCLUDING REMARKS

The paper is associated with the theoretical modelling of the structural problems of the stars. The approach can be utilized to obtained physical parameters related to the equilibrium structures of differentially rotating gaseous spheres having non-uniform masses. Suitable combinations of the constants involve in the law of differential rotations, provide a large number of stellar models of different natures. The method can be conveniently incorporated in any existing computer code, developed for computing the equilibrium structures of realistic polytropic models of stars. The method can also be utilized to determine the equilibrium structures of non-rotating stars $(b_1 = b_2 = b_3 = 0)$ as well as stars having solid body rotation (on using at least one parameter non-zero i.e. b_1, b_2, b_3). If the value of variable t is chosen as $t = 1$, structures of different stellar model having uniform masses, can be calculated. Since, the expressions related to the structures of stellar models are obtained in series form, therefore, our approach should not be applied to rapidly rotating stars.

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